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# HOW TO DRAW A PLANAR GRAPH ON A GRID

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Answering a question of Rosenstiehl and Tarjan, we show that every plane graph with $n$ vertices has a Fáry embedding (i.e., straight-line embedding) on the $2n-4$ by $n-2$ grid and provide an $O(n)$ space, $O(n \log n)$ time algorithm to effect this embedding. The grid size is asymptotically optimal and it had been previously unknown whether one can always find a polynomial sized grid to support such an embedding. On the other hand we show that any set $F$, which can support a Fáry embedding of every planar graph of size $n$, has cardinality at least $n+(1-o(1))\sqrt{n}$ which settles a problem of Mohar.

## 1. Introduction

The theorem of I. Fáry [4] shows that every plane graph has an embedding (drawing) in which the edges are straight line segments and the vertices are points in the plane. An embedding of this sort will be called a *Fáry embedding*. Starting with the paper of Tutte in 1963 there have been many algorithms offered for constructing a Fáry embedding ([20], [2], [16]). All present algorithms for Fáry embedding a plane graph exhibit several drawbacks. These drawbacks are: (1) that they require high precision real arithmetic relative to the size of the graph and (2) vertices tend to bunch together in the sense that the ratio of the smallest distance to the largest distance is unreasonably small. This means that: (1) for a graph of moderate size it is not possible to execute the algorithm and (2) even if it were, it would not be possible to view the resulting drawing on a terminal screen.

In fact, it has been an open question whether or not every planar graph of size $n$ has a Fáry embedding on a grid of side length bounded by $n^k$ for some fixed $k$ [17]. Questions, such as this, about how compactly graphs can be embedded on grids are related to the problems of VLSI layout design ([12], [21], [22]). Theorem 1 gives a positive answer (which is asymptotically sharp) to this question and its proof provides an algorithm constructing such an embedding.

**Theorem 1.** *Any plane graph with $n$ vertices has a Fáry embedding on the $2n-4$ by $n-2$ grid.*

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It can be shown that any Fáry embedding of a nested sequence of $\frac{n}{3}$ triangles on a grid requires a grid of size at least $\left(\frac{2}{3}n-1\right)\times\left(\frac{2}{3}n-1\right)$.

Since the Hopcroft—Tarjan planarity testing algorithm [11] outputs a topological embedding of a planar graph, we shall assume that any graph we look at has already passed that filter and comes equipped with such an embedding. A *maximal* plane graph is one which cannot have any additional edges without destroying its planarity. Such a graph is also called *triangulated* since all the faces are triangles. Since every planar graph can be triangulated by adding additional (dummy) edges, it suffices to prove Theorem 1 for maximal planar graphs.

We prove Theorem 1 in Section 3 by presenting an algorithm which takes an arbitrary maximal plane graph and an arbitrary triangular face and then outputs the embedding with the given triangle as the exterior face. Our proof is based on a general construction called the *canonical representation of plane graphs* (Section 2), which provides a suitable ordering of the vertices so that we can inductively Fáry embed the graph induced by the first $k$ vertices on a grid and then by moving some of the vertices in this embedding in a controlled way we are able to add the next vertex and still have a Fáry embedding. On the face of it this algorithm has at least quadratic time complexity. The speedup to $O(n \log n)$ is obtained by not actually performing each embedding but instead storing all the information needed in a single permutation which can be constructed in time $O(n \log n)$. Then we will make $O(n)$ queries of this permutation of the following form: for indices $i$ and $j$, how many $k$, $i \leqq k \leqq j$ precede $i$ in the permutation. The answers to the queries are then used to find the coordinates of the embedded vertices. These queries can be interpreted as rectangle range queries on a set of $2n$ points derived from the permutation and using a data structure of Chazelle [1] which uses linear space, $O(n \log n)$ preprocessing time, the queries can each be executed in time $O(\log n)$.

The canonical representation of plane graphs is a useful tool to establish the existence of Fáry embeddings with special geometric properties by easy inductional arguments. Some examples are Propositions 1, 2 and 3.

**Proposition 1.** *Given a maximal planar graph G and a face uvw, there is a labelling of the vertices,* $v_1=u, v_2=v, v_3, \ldots, v_n=w$ *and a Fáry embedding f such that the convex hull of* $\{f(v_1), f(v_2), f(v_3), \ldots, f(v_k)\}$ *is the same as the convex hull of* $\{f(v_1), f(v_2), f(v_k)\}$ *for* $k=4, \ldots, n$.

**Proposition 2.** *Given a maximal planar graph G and a face uvw, there is a labelling of the vertices,* $v_1=u, v_2=v, v_3, \ldots, v_n=w$ *and a Fáry embedding f such that the convex hull of* $\{f(v_1), f(v_2), f(v_3), \ldots, f(v_k)\}$ *is the same as the convex hull of* $\{f(v_{k-2}), f(v_{k-1}), f(v_k)\}$ *for* $k=4, \ldots, n$.

**Proposition 3.** *Given a maximal planar graph G and a face uvw, there is a labelling of the vertices,* $v_1=u, v_2=v, v_3, \ldots, v_n=w$ *and a Fáry embedding f such that the boundary of the convex hull of* $\{f(v_1), f(v_2), f(v_3), \ldots, f(v_k)\}$ *is a cycle in G and* $f(v_{k+1})$ *is not contained in the convex hull of* $\{f(v_1), f(v_2), f(v_3), \ldots, f(v_k)\}$.

Another consequence of the canonical representation of plane graphs is the following lemma which plays a key role in a new characterization of planar graphs due to Schnyder [18].

**Proposition 4.** *Given a maximal planar graph $G$ with exterior face $uvw$, there is a labelling of the angles of the internal triangles with labels* 1, 2 *and* 3 *such that*

(i) *each triangle has labels* 1, 2 *and* 3 *in counterclockwise order;*
(ii) *all angles at $u$, $v$ and $w$ are labelled* 1, 2 *and* 3, *respectively;*
(iii) *around each internal vertex the angles of each label appear in a single block.*

Theorem 1 gives an asymptotically sharp bound on the size of the smallest grid that will support all planar graphs of size $n$. Even though a grid is a natural set on which to embed graphs, it also makes sense to drop the restriction to a grid and ask for bounds on the size of a set which supports all planar graphs of size $n$. Last year Bojan Mohar [13] asked whether or not there exists a set $F$ of $n$ points in the plane which supports every planar graph with $n$ vertices (a set $F$ *supports* a simple planar graph if there exists an injective map $f\colon V(G)) \to F$ such that the segments $[f(a), f(b)]$ and $[f(c), f(d)]$ do not cross if $[a, b]$ and $[c, d]$ are edges of $G$). $F$ is called *universal* for a set of planar graphs if it supports all graphs in the set. Thus $\mathbf{R}^2$ is universal for all planar graphs by the theorem of Fáry. By Theorem 1 the $n-2$ by $2n-4$ grid is universal for all planar graphs with $n$ vertices, and Mohar is asking whether there is a universal set of size $n$ for all planar graphs with $n$ vertices. We give a negative answer to this in Section 5 by proving;

**Theorem 2.** *If $F$ is universal for planar graphs with $n$ vertices then*

$$|F| > n + (1 - o(1))\sqrt{n}.$$

A planar graph which can be obtained from a simple cycle by adding some of its internal diagonals is called *outerplanar.*

**Proposition 5.** *Every set of $n$ points in the plane, in general position, supports every outerplanar graph with $n$ vertices. Moreover, this property characterizes the outerplanar graphs.*

In other words, any $n$-element point set in general position is universal for a rather large family of planar graphs of size $n$. This may suggest that a fairly small point set can be universal for all planar graphs of size $n$.

## 2. The canonical representation of plane graphs

The aim of this section is to describe a canonical way of constructing a plane graph, which will be a basic tool of our investigations in the rest of this paper, and also provides easy proofs of generalizations of Fáry's theorem (Propositions 1, 2 and 3).

The following simple observation will be essential for our purposes.

**Lemma.** *Let $G$ be a simple planar graph embedded in the plane and $u=u_1, u_2, \dots, u_k=v$ be a cycle of $G$. Then there exists a vertex $w'$ (resp. $w''$) on the cycle, different from $u$ and $v$ and not adjacent to any inside chord (resp. outside chord).*

**Proof.** If the cycle has no inside (resp. outside) chords, then there is nothing to prove. Otherwise, let $(u_i, u_j), j > i+1$ be an inside (outside) chord such that $j-i$ is minimal. Then $u_{i+1}$ cannot be adjacent to any inside chord of the cycle $u_i, u_{i+1}, \dots, u_j$, by minimality. Nor can it be adjacent to any other inside (outside) chord of the original cycle, by planarity. ∎

Now we are in the position to establish the following.

**Canonical representation lemma for plane graphs.** *Let $G$ be a maximal planar graph embedded in the plane with exterior face $u$, $v$, $w$. Then there exists a labelling of the vertices $v_1=u, v_2=v, v_3, \dots, v_n=w$ meeting the following requirements for every $4 \leq k \leq n$.*

(i) *The subgraph $G_{k-1} \subseteq G$ induced by $v_1, v_2, \dots, v_{k-1}$ is 2-connected, and the boundary of its exterior face is a cycle $C_{k-1}$ containing the edge $uv$;*

(ii) *$v_k$ is in the exterior face of $G_{k-1}$, and its neighbors in $G_{k-1}$ form an (at least 2-element) subinterval of the path $C_{k-1}-uv$.* (See Fig. 1.)

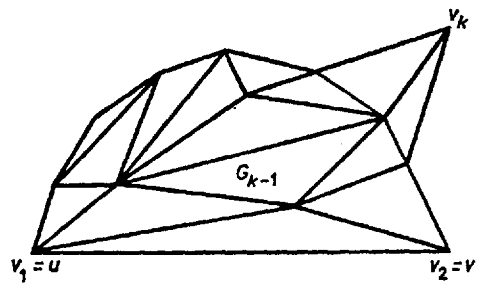


*Fig. 1*

**Proof.** The vertices $v_n, v_{n-1}, \dots, v_3$ will be defined by reverse induction.

Set $v_n=w$, and let $G_{n-1}$ denote the subgraph of $G$ after deleting $v_n$. By the maximality of $G$, the neighbors of $w$ form a cycle $C_{n-1}$ containing $uv$, and this cycle is the boundary of the exterior face of $G_{n-1}$.

Let $i<n$ be fixed, and assume that $v_k$ has already been determined for every $k>i$ such that the subgraph $G_{k-1}$ induced by $V(G) \setminus \{v_k, v_{k+1}, \dots, v_n\}$ satisfies conditions (i) and (ii). Let $C_{k-1}$ denote the boundary of the exterior face of $G_{k-1}$. Applying the Lemma to the cycle $C_i$ in $G_i$, we obtain that there is a vertex $w' \in C_i$ different from $u$ and $v$ and not adjacent to any chord of $C_i$. (Observe that $C_i$ has no exterior chords.) Letting $v_i=w'$, the subgraph $G_{i-1}$ induced by $V(G) \setminus \{v_i, v_{i+1}, \dots, v_n\}$ obviously meets the requirements. ∎

Proposition 1 now follows almost immediately.

**Proof of Proposition 1.** Let $v_1=u$, $v_2=v, v_3, \dots, v_n=w$ be the canonical labelling of the vertices of $G$, as described above. We will define $f(v_k)$, $1 \leq k \leq n$ by induction on $k$.

Set $f(v_1)=(0,0)$, $f(v_2)=(2,0)$, $f(v_3)=(1,1)$. Assume that $f(v_i)$ has already been determined for $i=1, 2, \dots, k-1$ such that $f$ is a Fáry embedding of $G_{k-1}$ with

$$\text{conv}\,\{f(v_1), f(v_2), \dots, f(v_i)\} = \text{conv}\,\{f(v_1), f(v_2), f(v_i)\}, \quad 3 \leq i \leq k-1.$$

We want to extend it to an embedding of $G_k$.

Let $u=w_1, w_2, \dots, w_m=v$ denote the vertices of $C_{k-1}$ in the order as they appear along the boundary of the exterior face of $G_{k-1}$. Let $x(w_j)$ and $y(w_j)$ denote the $x$-coordinate and $y$-coordinate of $f(w_j)$, respectively. Suppose, by induction,

$$\text{(iii)} \qquad x(w_1) < x(w_2) < \dots < x(w_m),$$

$$y(w_j) > 0 \quad \text{for} \quad 3 \leq j \leq m.$$

By property (ii) of the canonical labelling, $v_k$ is connected to $w_p, w_{p+1}, \ldots, w_q$ for some $1 \leqq p < q \leqq m$.

Let us fix any number $x^*$ between $x(w_p)$ and $x(w_q)$, and set $f(v_k)=(x^*, y^*)$ for some $y^*>0$. It is now clear that, if $y^*$ is sufficiently large, then we obtain a Fáry embedding of $G_k$ with the desired properties. Furthermore, our auxiliary hypothesis (iii) will remain true for the points of $C_k$. ∎

Propositions 2 and 3 can be proved in a similar way.

## 3. Drawing a plane graph on a grid

It suffices to prove Theorem 1 for maximal plane graphs. Let $G$ be such a graph with exterior face $u, v, w$, and let $v_1=u, v_2=v, v_3, \ldots, v_n=w$ be the canonical labelling of its vertices.

The idea of the proof is the following. Suppose that at step $k$ of our algorithm $G_k$ has already been Fáry embedded on the grid such that

(1) $v_1$ is at $(0,0)$, $v_2$ is at $(2k-4, 0)$;

(2) If $v_1=w_1, w_2, \ldots, w_m=v_2$ denote the vertices on the exterior face of $G_k$ (in the order of their appearance), and $x(w_i)$ denotes the $x$-coordinate of $w_i$, then

$$x(w_1) < x(w_2) < \ldots < x(v_m);$$

(3) The edges $[w_i, w_{i+1}]$, $1 \leqq i < m$, all have slopes $+1$ or $-1$.

Note that (3) implies that the Manhattan distance between any two vertices $w_i$ and $w_j$ of the exterior face is *even*. (The Manhattan distance of $(x, y)$ and $(x', y')$ is $|x-x'|+|y-y'|$.) Hence, if $i<j$, then the intersection of the line with slope $+1$ through $w_i$ and the line with slope $-1$ through $w_j$ is a *lattice point* $P(w_i, w_j)$.

Let $w_p, w_{p+1}, \ldots, w_q$ be the neighbors of $v_{k+1}$ in $G_{k+1}$ $(1 \leqq p < q \leqq m)$, (cf. part (ii) of the Canonical Representation Lemma). Then $P(w_p, w_q)$ is a good candidate for placing $v_{k+1}$, except that it may fail to see e.g., $w_p$ (see Fig. 2). To make sure that $P(w_p, w_q)$ sees all the points $w_p, w_{p+1}, \ldots, w_q$, we shall deform the embedding (drawing) to guarantee that the slope of the edge $[w_p, w_{p+1}]$ is $<1$ and the slope of $[w_{q-1}, w_q]$ is $>-1$ while the slopes of all other edges of the exterior face of $G_k$ remain the same. One way to ensure this is to move $w_{p+1}, w_{p+2}, \ldots, w_m$ one unit to the right and then to move $w_q, w_{q+1}, \ldots, w_m$ an additional unit to the right. However, to keep a Fáry embedding, it may be necessary to move some other vertices (not on the exterior face) as well. Though it is difficult to know globally which set of points has to move together with a given exterior vertex, there is an elegant way to define such sets recursively at each step of our algorithm.

To realize this goal, assume that for each vertex $w_i$ on the exterior face of $G_k$ we have already defined a subset $M(k, w_i) \subseteqq V(G_k)$ such that

(a) $w_j \in M(k, w_i)$ iff $j \geqq i$;

(b) $M(k, w_1) \supset M(k, w_2) \supset \ldots \supset M(k, w_m)$;

(c) For any nonnegative numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, if we sequentially translate all vertices in $M(k, w_i)$ with distance $\alpha_i$ to the right $(i=1, 2, \ldots, m)$, then the embedding of $G_k$ remains a Fáry embedding. (Note that many vertices will move several times; e.g., all points in $M(k, w_i) \setminus M(k, w_{i+1})$ will be translated by $\alpha_1+\alpha_2+\ldots+\alpha_i$.)

For $k=3$ these conditions are met by the Fáry embedding $v_1 \to (0, 0)$, $v_2 \to (2, 0)$, $v_3 \to (1, 1)$ and by the sets $M(3, v_1) = \{v_1, v_2, v_3\}$, $M(3, v_2) = \{v_2, v_3\}$, $M(3, v_3) = \{v_3\}$.

Apply condition (c) with $\alpha_{p+1} = \alpha_q = 1$ and all other $\alpha_i = 0$ to find a new Fáry embedding of $G_k$. The Manhattan distance between $w_p$ and the new location of $w_q$ is still even, thus we can place $v_{k+1}$ at the intersection of the lines with slope $+1$ and $-1$ through $w_p$ and the new location of $w_q$, respectively. Conditions (1), (2) and (3) will trivially remain true for this new embedding of $G_{k+1}$. (See Fig. 3.)

The vertices of the exterior face of $G_{k+1}$ are

$$u = w_1, w_2, \ldots, w_p, v_{k+1}, w_q, \ldots, w_m = v.$$

For each member $z$ of this sequence we have to define a set $M(k+1, z) \subseteq V(G_{k+1})$. Let

$$M(k+1, w_i) = M(k, w_i) \cup \{v_{k+1}\} \quad \text{for} \quad i \leqq p,$$

$$M(k+1, v_{k+1}) = M(k, w_{p+1}) \cup \{v_{k+1}\},$$

$$M(k+1, w_j) = M(k, w_j) \quad \text{for} \quad j \geqq q.$$

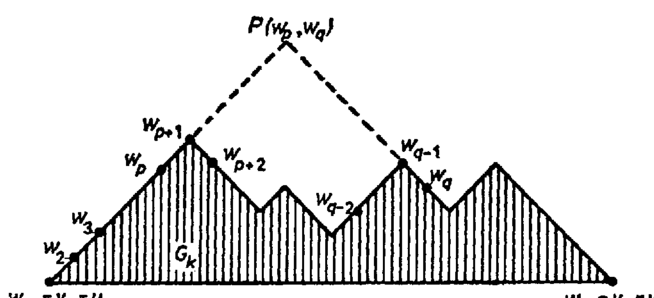


*Fig. 2*

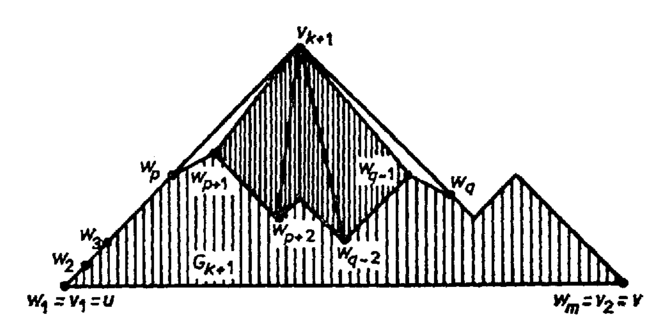


*Fig. 3*

It is obvious, by induction, that these sets have properties (a) and (b).

To check that property (c) remains true as well, fix a sequence of nonnegative numbers $\alpha(w_1), \dots, \alpha(w_p), \alpha(v_{k+1}), \alpha(w_q), \dots, \alpha(w_m)$. For all $z$ on the exterior face of $G_{k+1}$ translate the sets $M(k+1, z)$ with distance $\alpha(z)$ to the right. Observe that after this motion the part of $G_{k+1}$ below the polygon $w_1 w_2 \dots w_m$ (i.e., $G_k$) remains Fáry embedded (by condition (c) applied to $G_k$ with $\alpha_1=\alpha(w_1), \dots, \alpha_p=\alpha(w_p), \alpha_{p+1}= =1+\alpha(v_{k+1}), \alpha_q=1+\alpha(w_q), \alpha_{q+1}=\alpha(w_{q+1}), \dots, \alpha_m=\alpha(w_m)$ and every other $\alpha_i=0$). On the other hand, it is easy to see that the part of $G_{k+1}$ above $w_1 w_2 \dots w_m$ (i.e., $v_{k+1}$ and the upper contour of $G_k$) remains Fáry embedded, too, since during the motion the (darkly shaded) subgraph induced by $w_{p+1}, w_{p+2}, \dots, w_{q-1}$ and $v_{k+1}$ moves rigidly (to a distance $\alpha(w_1)+\dots+\alpha(w_p)+\alpha(v_{k+1})$).

The final output of our algorithm is a Fáry embedding $f$ of $G_n=G$ satisfying conditions (1), (2) and (3) with $k=n$. This immediately implies that every point of $G$ is embedded in some lattice point of the triangle determined by $f(v_1)= =f(u)=(0,0)$, $f(v_2)=f(v)=(2n-4, 0)$ and $f(v_n)=f(w)=(n-2, n-2)$. This completes the proof of Theorem 1. ∎

## 4. Outline of an $O(n \log n)$ algorithm for drawing a maximal planar graph on a grid

Given a planar graph $G$ we might as well assume that $G$ is maximal, since, in linear time dummy edges can be added to make it so [16]. Let $v_1 v_2 w$ denote the exterior face of $G$. We further label all vertices as (*a*) not yet visited, (*b*) visited once or (*i*) visited more than once and the visited edges form $i$ connected components in the circular order of all edges adjacent to this vertex. These labels are updated after we choose vertex $v_{k+1}$. We visit each neighbor of $v_{k+1}$ along the edge connecting them. Suppose that $v$ is such a vertex. If $v$ has label (*a*), label (*b*) replaces label (*a*). If $v$ has label (*b*) and the edge from $v_{k+1}$ is adjacent to a previous edge, in the circular order of edges at $v$, along which $v$ was visited, label (1) replaces label (*b*) and if not, label (2) replaces label (*b*). Finally if $v$ has label ($j$) and the left and right neighbors of the edge from $v_{k+1}$ to $v$ have already been traversed then label ($j-1$) replaces label $j$. If only one of these edges has been traversed then the label is unchanged and if neither has been traversed then label ($j+1$) replaces label ($j$). It is clear that the label ($j$) on $v$ means that the edges already traversed and incident to $v$ are composed of $j$ intervals in the circular order of edges at $v$. It is easy to see that if $k+2<n$ then there is a vertex with label (1) different from $w$, and this can be chosen as $v_{k+2}$. Since there are only a linear number of edges, we find the order to insert vertices in linear time.

We define a sequence of permutations inductively as follows. $\pi_2=(1, 2)$. Suppose $\pi_k$ is defined and the vertex $v_{k+1}$ is adjacent (in counterclockwise order) to the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_j}$ in $G_k$. Then we generate $\pi_{k+1}$ by inserting $k+1$ just to the left of $i_2$ and $n+k+1$ just to the left of $i_j$ in the permutation $\pi_k$. Clearly $\pi_n$ can be constructed in time $O(n \log n)$. It is clear, identifying the vertex $v_j$ with the index $j$, that

$$M(k, v_i) = \{j \mid j \leqq n, j \text{ does not precede } i \text{ in } \pi_k\} =$$

$$= \{j \mid i \leqq j \leqq k, j \text{ does not precede } i \text{ in } \pi_n\}.$$

Suppose that $j<k$ and $v_j$ has coordinates $(x_j(k), y_j(k))$ when $v_k$ is placed

on the grid, i.e., at the $k$th step of the algorithm of the last section. Then $y_j(k)=$ $=y_j(j)$ and $x_j(k)=x_j(j)+\sigma(j,k)$ where

$$\sigma(j,k) = \left|\{i|j<i, i \text{ precedes } j \text{ in } \pi_k\}\right| =$$
$$= \left|\{i|j<i\leqq k, i \text{ precedes } j \text{ in } \pi_n\}\right| + \left|\{i|j<i\leqq k, i+n \text{ precedes } j \text{ in } \pi_n\}\right|.$$

It follows that we can find $(x_k(k), y_k(k))$ in constant time given $(x_j(j), y_j(j))$ and $\sigma(j,k)$ for $v_j$ the left-most neighbor of $v_k$ and for $v_j$ the right-most neighbor of $v_k$ in $G_{k-1}$. Then in constant time we find the embedded coordinates of $v_k$, $(x_k(n), y_k(n))$. Thus, the entire algorithm runs in time $O(nT)$, where $T$ is the time to calculate any $\sigma(j,k)$. Finally, let $S$ be the set of points

$$\left\{(t, \pi_n^{-1}(t)), (t, \pi_n^{-1}(n+t))|3 \leqq t \leqq n\right\}.$$

It is evident that $\sigma(j,k)=|S\cap R(j,k)|$ where $R(j,k)$ is the rectangle,

$$R(j,k) = \{(x,y)|j+1 \leqq x \leqq k,\ 1 \leqq y \leqq \pi_n^{-1}(j)\}.$$

Hence it follows from [1] that $T=\log n$. ∎

## 5. Lower bound on the size of a set which supports all planar graphs

Let $G_k$ be a fixed maximal planar graph with $k$ vertices and $2k-4$ triangular faces. Given any natural numbers $n_i$, $1\leqq i\leqq 2k-4$ with $\sum_{i=1}^{2k-4} n_i=n-k$, let $G_k(n_1, n_2, \ldots, n_{2k-4})$ be a fixed maximal planar graph on the vertices $\{P_1, P_2, \ldots, P_n\}$ whose restriction to $\{P_1, P_2, \ldots, P_k\}$ is $G_k$ and has $n_i$ vertices in the face $f_i$. Since $G_k$ and $G_k(n_1, n_2, \ldots, n_{2k-4})$ are maximal, Steinitz's theorem shows that they have unique planar maps. There are $\binom{n+k-5}{2k-5}$ of these graphs. Now suppose that $F$, a subset of $\mathbf{R}^2$ supports all these graphs, and fix an embedding $f_G\colon V(G)\to F$ for each of them. There are at most $|F|^k$ ways to embed $(P_1, P_2, \ldots, P_k)$, hence the embedding of at least $\binom{n+k-5}{2k-5}/|F|^k$ of these graphs is the same on the vertices $\{P_1, P_2, \ldots, P_k\}$. On the other hand a given embedding of $G_k$ on $F$ can be extended to at most $\binom{|F|-n+2k-5}{2k-5}$ of our graphs. Hence,

$$\binom{|F|-n+2k-5}{2k-5} > \binom{n+k-5}{2k-5}/|F|^k.$$

From this, by elementary calculations we obtain

$$\left(\frac{(|F|-n+2k)^2}{n}\right)^k \left(\frac{|F|}{n}\right)^k > n^{-5}\left(1-\frac{k}{n}\right)^{2k}.$$

Now choosing $k=\frac{\varepsilon}{4}\sqrt{n}$ we contradict $|F|<n+(1-\varepsilon)\sqrt{n}$ for $n$ sufficiently large, and Theorem 2 follows. ∎

## 6. Universal sets for planar graphs

A universal configuration for planar graphs of size $n$ is a family of sets $F_1, F_2, \ldots, F_n$ such that every planar graph of size $n$ has an embedding on the union of the $F$'s in such a way that no two vertices are mapped to the same $F_i$. The question of B. Mohar can be stated in these terms as asking whether there is a universal configuration in which the sets $F_i$ are singletons. Even though the answer to this question is no (Theorem 2), a slight modification of the last section shows that there are universal configurations in which the sets $F_k$ are arbitrarily small.

**Proposition 6.** *Given any* $\varepsilon > 0$, *the* $n$ *horizontal intervals of length* $\varepsilon$ *centered at* $(0, i)$, $i = 1, 2, \ldots, n$ *support every planar graph on* $n$ *vertices.*

As in Section 3 we will embed the plane graph $G$ on a lattice incrementally by the subgraphs $G_3, G_4, \ldots, G_n = G$. This time, however, we shall place the vertex $v_k$ at a point with $y$ coordinate $k$. In order to satisfy this constraint we will give up control over the size of the $x$ coordinate. Suppose that we have embedded $G_k$ satisfying these conditions, and we have defined the sets $M(k, w_i)$ having properties (a), (b), (c) in Section 3. To insert $v_{k+1}$, we sequentially move $M(k, w_{p+1}), M(k, w_{p+2}), \ldots, M(k, w_q)$ by distances $\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_q$ to the right so that the slopes of all edges of the polygon $w_p w_{p+1} \ldots w_q$ become very small. So small that $w_p, w_{p+1}, \ldots, w_q$ are visible from a point with $y$ coordinate $k+1$ and with $x$ coordinate between those of $w_p$ and $w_q$. We place $v_{k+1}$ at such a point and update $M(k+1, z)$ as in Section 3. Repeating this process, we have an embedding of $G$ with $v_k$ at $(x_k, k)$. The affine transformation $y' = y$, $x' = (\varepsilon / \max x_k) x$ shows that $F_i = [(0, i), (\varepsilon, i)]$, $(i = 1, \ldots, n)$ is a universal configuration and Proposition 6 is proved.

## 7. Remarks

The algorithm implicit in the proof of Theorem 1, as well as another which introduces new edges on the exterior face which may be horizontal at the time of insertion rather than insisting that they have slopes 1 or $-1$ has been implemented by Nejia Assila. This second version, has the advantage that the graph may have a Fáry embedding on a considerably smaller grid. Figures 4a and 4b show the output of each algorithm on the same graph.

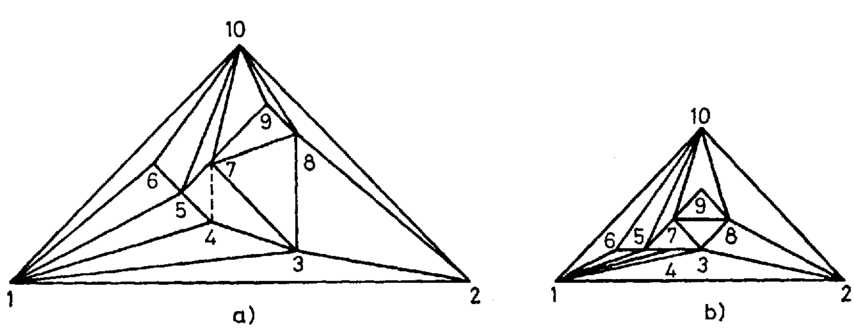


*Fig. 4*

For $f(n)$, the smallest cardinality of a set which supports all planar graphs, Theorems 1 and 2 show that

$$n+(1-o(1))\sqrt{n} \leqq f(n) < n^2.$$

An interesting open problem is to tighten these bounds. A possible approach could be to split a planar graph into outerplanar subgraphs and attempt to match the embeddings guaranteed by Proposition 5. The simple inductive proof of Proposition 5, found independently, by P. Gritzmann and B. Mohar ([9]) and the authors, is left to the reader.

Our drawing algorithm outlined in Section 5 uses $O(n \log n)$ time and linear space except for the step where we use the linear time triangulation algorithm of Read [16] for planar graphs which in fact uses quadratic space. Here we sketch a linear time, linear space algorithm which may replace Read's method.

We start with a connected planar graph $G$, where the neighbors of each vertex $v$ are given in their circular (counterclockwise) order around $v$. We visit the vertices $v$ successively. At each vertex $v$, we read the list of its neighbors. Let $w$ immediately follow $u$ on this list. If $u$ does not immediately follow $v$ in the circular order around $w$, then we insert the edge $uw$ in its correct position at $u$ and at $w$. That is, $u$ will follow $v$ at $w$, and $v$ will follow $w$ at $u$. This insertion is done in constant time. Moreover, we label this edge "new". (Note that $u$ and $w$ may already be connected in the graph by an edge in another position!) We continue in this way with all the neighbors of $v$. Then, after the passage of linear time we obtain a triangulated planar multigraph $G'$ which contains $G$ as a subgraph, and it is easy to see that each new edge whose endpoints $(i,j)$ are connected by another edge of $G'$ is an internal diagonal of a unique quadrilateral $ii'jj'$ whose other diagonal $(i'j')$ is not an edge of $G'$. Now we replace the edge $ij$ by the edge $i'j'$ in its correct position. These replacements can be done by first sorting the edges of $G'$ lexicographically (according to their endpoints) and then doing one pass through this sorted list. (Note that $G'$ has $3n-6$ edges and the sorting can be done in linear time by radix sort.) Thus, after a little more linear time, we obtain a triangulation of $G$ using only linear space.

Finally, we suspect that a linear time algorithm exists to Fáry embed any planar graph on a linear sized grid. A weaker, but equally important, question is whether the algorithm can be improved dynamically. Is it possible, given an embedding, to insert a new vertex in linear time?

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**Added in proof.** M. Chrobak and T. Payne have recently proved that our drawing algorithm described in Section 4 can in fact be executed in linear time, by handling the data more carefully (see M. Chrobak and T. H. Payne, A Linear-time Algorithm for Drawing a Planar Graph on a Grid, Manuscript, University of California at Riverside (1988)). M. Chrobak also managed to improve the lower bound in our Theorem 2 to $1.04n$ (personal communication). Another exciting new development is that W. Schnyder showed that any plane graph can be Fáry embedded on an

$n-1$ by $n-1$ grid (see W. Schnyder, Embedding Planar Graphs on the Grid, Proceedings, of the First ACM-SIAM Symposium on Discrete Algorihms, pp. 138—147, (1990)).

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